

NONITERATIVE SPLICING IN SOLVING HEAT-CONDUCTION PROBLEMS  
IN COMPLEX THREE-DIMENSIONAL STRUCTURES

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UDC 536.2:518.61

An economic numerical method is developed for the solution of heat-conduction problems in three-dimensional structures.

In calculating temperature fields in complex three-dimensional structures, there are considerable difficulties associated with the large volume of information to be analyzed and the complex structure of the system of equations.

The basic approach to the solution of this problem is to divide the structure into subregions and to divide the computational algorithm into a series of homogeneous problems for the subregions, with a definite algorithm for splicing the solutions obtained in the subregions to yield a general solution for the whole structure.

The splicing method must: 1) ensure the same order of approximation of the physical boundary conditions at the common boundaries of the subregions as within the subregions; 2) not impose additional constraints on the stability condition of the general solution; 3) ensure solution of the problems in the subregions by economic methods; 4) be economic in terms of the number of operations; 5) impose no constraints on the configuration of the system of subregions (the structure) as a whole; 6) be technologically expedient from the viewpoint of program realization and impose no constraints on: a) the order of calculation of the subregions; b) the parameters of their grid division; c) the order of calculation with coordinate division of the solution into subregions.

The solution of this problem in the complete volume has only been obtained for rod structures (one-dimensional systems) [1]. For two- and three-dimensional systems, there are no such solutions. In [2, 3], splicing algorithms were proposed for the case of joining two or more plates with a common longitudinal coordinate. In [4], various particular methods of solving some more complex plate systems were considered. In the present work, a general splicing method is developed for two- and three-dimensional subregions.

Suppose that there is a structure in the form of a system of arbitrarily connected rectangular plates and parallelepipeds. Heat propagation in this structure is described by a system of heat-conduction equations for each subregion, related by the splicing conditions: continuity (taking account of contact heat transfer) of the temperature field and energy conservation at the boundaries between the subregions. Boundary conditions of any type may be specified at the free boundaries, and internal sources whose intensity may depend both on the time and spatial coordinates and on the temperature may be specified within the subregions.

The numerical solution of the problem is constructed on the basis of the method of elementary heat balances. In each subregion, a different independent grid for division into elements is introduced, and different systems of node points at which the splicing conditions must be satisfied are introduced at the boundary lines and surfaces between the subregions.

One of the basic problems is to choose an economic method of solving the system of difference equations in the subregion. The known economic schemes of coordinate division do not satisfy the above requirements, since they are not symmetric relative to the order of calculation with respect to the coordinates. In connection with this, a division scheme is proposed, assuming complete independence of the heat-propagation processes with respect to each coordinate, within the limits of a single time step. The equation

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Translated from *Inzhenerno-Fizicheskii Zhurnal*, Vol. 56, No. 3, pp. 478-483, March, 1989. Original article submitted April 18, 1988.

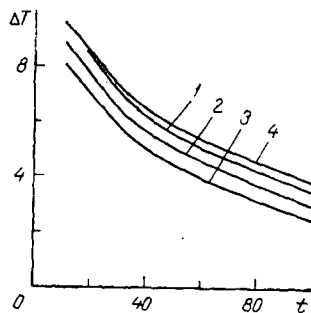


Fig. 1

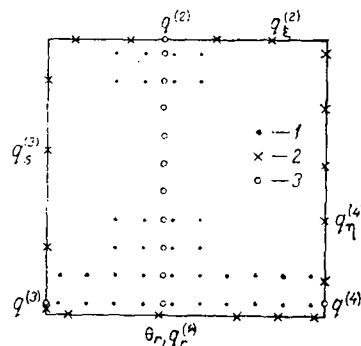


Fig. 2

Fig. 1. Error of numerical solution: 1) scheme without division; 2) usual division scheme; 3) scheme in Eq. (1); 4) scheme in Eq. (1), taking account of splicing algorithm.  $\Delta T$ , K;  $t$ , sec.

Fig. 2. Splicing solutions in inconsistent grids: 1) centers of elements; 2) node points; 3) points for the formation of the splicing conditions at point  $r$ .

$$c\rho \frac{\partial T}{\partial t} = LT + f(x, t, T), \quad x = x_1, \dots, x_\beta,$$

where

$$LT = \sum_{\alpha=1}^{\beta} L_{\alpha} T, \quad L_{\alpha} T = \frac{\partial}{\partial x_{\alpha}} \left( \lambda \frac{\partial T}{\partial x_{\alpha}} \right),$$

in the interval  $(t_{k-1}, t_k)$  is replaced by the following system of equations

$$c\rho \frac{\partial U_{\alpha}}{\partial t} = L_{\alpha} U_{\alpha} + \frac{1}{\beta} f(x, t, U_{\alpha}), \quad \alpha = 1, \dots, \beta, \tag{1}$$

$$U_{\alpha}(x, t_{k-1}) = T_{k-1},$$

$$T_k = \sum_{\alpha=1}^{\beta} U_{\alpha} - (\beta - 1) T_{k-1}.$$

The scheme in Eq. (1) is tested in trial problems, the results for one of which are shown in Fig. 1. A square plate is considered; heat-transfer conditions are specified at two adjacent boundaries ( $T_b = 1200$  K;  $\alpha = 140$  W/m<sup>2</sup>·K), while the other two are assumed to be heat-insulated. The number of elements with respect to each coordinate is the same: 10; the calculation step is 10 sec. In Fig. 1, the errors for the most heated corner element in the numerical solution of the problem are shown. Note that, for the given symmetric problem, the solution obtained is symmetric, except for the discharge behavior of the computer. Thus, the scheme in Eq. (1) satisfies all the above requirements.

To ensure splicing of the solutions obtained in the subregions, it is necessary to determine the relation between the heat fluxes and temperatures at the boundaries between subregions for the  $k$ -th time step from the temperature values of the subregion elements known at the  $(k - 1)$ -th step and from data on the thermal perturbations at its free boundaries.

To this end, a two-dimensional rectangular region (plate) is considered. The lower and upper boundaries are denoted by numbers 1 and 2, respectively, and the left-hand and right-hand boundaries by 3 and 4. Quantities belonging to a particular boundary are denoted by the corresponding superscript. The subregion is divided into  $I$  and  $J$  elements in the directions of the coordinate axes. For the sake of simplicity, it is assumed, first of all, that the division of the subregions and the system of node points are consistent. The temperature of the elements in the  $(k - 1)$ -th step is denoted by  $T_{ijk-1}$ ,  $i = 1, \dots, I$ ,  $j = 1, \dots, J$ , and the temperatures and heat fluxes at node points in the  $k$ -th step, correspondingly, by  $\theta_r^{(\gamma)}$  and  $q_r^{(\gamma)}$ ,  $r = 1, \dots, I$ ,  $r = 1, \dots, J$ , where  $\gamma$  is the number of the boundary.

Assuming that, at the  $(k - 1)$ -th time step, the temperatures of all the elements are known, and that boundary conditions of the second kind - i.e., heat fluxes at the node points - are specified at the boundaries, the temperatures of the near-boundary elements at the  $k$ -th step is expressed in terms of the heat fluxes at node points of the boundaries. To this end, in accordance with the proposed division scheme, the problem of conversion from one time step to another is divided into a series of one-dimensional equations for a chain of rod elements in the direction of the coordinate axes. Using a system of difference equations for rods perpendicular to the boundary, for example, the first, and direct parametric fitting from the opposite (second) boundary, the following expression is obtained for the temperature of near-boundary elements of the first boundary at the  $k$ -th step as a result of heat leakage in the direction perpendicular to the boundary

$$U_{i1} = R_i q_i^{(1)} + S_i q_i^{(2)} + P_i \quad (2)$$

Direct and inverse parametric fitting with respect to a near-boundary rod leads to the following expression for the temperature of the near-boundary elements at the  $k$ -th step as a result of heat leakage parallel to the boundary

$$U'_{i1} = R'_i q_i^{(3)} + S'_i q_i^{(4)} + P_i \quad (3)$$

Finally, from Eqs. (1)-(3), the temperature of the near-boundary elements of the first boundary is

$$T_{i1} = U_{i1} + U'_{i1} - T_{i1k-1} = R_i q_i^{(1)} + S_i q_i^{(2)} + R'_i q_i^{(3)} + S'_i q_i^{(4)} + (P_i + P'_i - T_{i1k-1}) \quad (4)$$

Using a difference approximation for the heat flux at the boundary

$$q_i^{(1)} = -\frac{2\lambda}{h} (T_{i1} - \theta_i^{(1)}) \quad (5)$$

and substituting  $T_{i1}$  from Eq. (4) into Eq. (5), the following relation between the temperature at node points of the first boundary and the boundary heat fluxes is obtained

$$\theta_i^{(1)} = \sum_r a_{ir} q_r + b_i \quad (6)$$

where  $r$  is the sequence of points associated, according to the division scheme adopted, with the  $i$ -th point of the first boundary;  $a_{ir}$ ,  $b_i$  are coefficients.

If any of the boundaries  $\gamma \neq 1$  is free, i.e., there are specified boundary conditions, the term with the corresponding  $q_r$  in Eq. (6) is replaced by its numerical value and is included in  $b_i$ .

Performing this procedure for all the subregion boundaries that are not free boundaries, a system of equations for the subregion is obtained

$$\bar{\theta} = A\bar{q} + \bar{B}, \quad (7)$$

where  $A$  is the coefficient matrix,  $\bar{\theta}$ ,  $\bar{q}$ ,  $\bar{B}$  are vectors.

Inverting the matrix  $A$ , an expression for the heat fluxes is obtained

$$\bar{q} = A^{-1}\bar{\theta} - A^{-1}\bar{B} \quad (8)$$

Determining Eq. (8) for the heat fluxes at node points for all the subregions and substituting the results into the energy-conservation equations at the boundaries between subregions at each of the  $M$  node points

$$\sum_{\eta} q_{m\eta} = 0, \quad m = 1, \dots, M,$$

the following system of equations for the temperature at the node points at the  $k$ -th time step is obtained

$$\sum_{\varepsilon} C_{m\varepsilon} \theta_{\varepsilon} = D_m, \quad m = 1, \dots, M. \quad (9)$$

Here  $\eta$ ,  $\varepsilon$  are the number of subregions connecting at the given point and the number of points of these subregions.

Solving Eq. (9), the temperature values at the node points are determined for the k-th time step. Then, from Eq. (8), the heat fluxes arriving at each subregion across the boundaries with other subregions are calculated.

Thus, all the boundary conditions for each subregion are determined and the temperature distribution there in the k-th step is calculated from the scheme in Eq. (1).

This method is tested by comparing the results of calculation for the heating of a plate divided into four ( $2 \times 2$ ) subregions with the data of "continuous" calculation of the plate. The case which is most significant for this method is that of calculation with the introduction of node points at heated boundaries, which was done in the above example with a plate for the same calculation parameters. The resulting errors are shown in Fig. 1 (curve 4). It is evident that the method does not produce more errors than the usual difference scheme without division (curve 1).

Now consider the case when the division of the subregions into elements is not consistent. In this case, independent systems of node points are introduced at all the boundaries between the subregions. The positions of the node points on opposite boundaries of the subregion are also inconsistent.

An interpolational relation between the temperatures and heat fluxes at the grid points associated with the system of points and elements introduced is established by the method of smooth execution, ensuring smoothness of the interpolation function.

The procedure for constructing the system in Eq. (7) for the node-point temperatures remains the same in broad outline, but the lack of agreement between the node points and the lines of the centers of the elements introduces some distinctive features. Above all, this is expressed in that Eq. (4) for the boundary element must be replaced by a similar expression for a point on the line of the centers of boundary elements opposite the given node point (Fig. 2).

A line perpendicular to the boundary is drawn from the given node point  $r$ . The heat flux at the point of intersection of this line with the opposite boundary is denoted by  $q^{(2)}$ . The temperatures in the  $(k - 1)$ -th step at the points of intersection of the perpendicular with the lines joining the centers of the elements are calculated by interpolation. Parametric fitting from the opposite boundary gives

$$U_{r1} = R_r q_r^{(1)} + S_r q^{(2)} + P_r \quad (10)$$

Using parametric interpolation,  $q^{(2)}$  is expressed in terms of  $q_\xi^{(2)}$  at the node points

$$q^{(2)} = \sum_{\xi} d_{\xi} q_{\xi}^{(2)}, \quad (11)$$

where  $d_{\xi}$  are interpolation coefficients.

Finally,

$$U_{r1} = R_r q_r^{(1)} + S_r \sum_{\xi} d_{\xi} q_{\xi}^{(2)} + P_r. \quad (12)$$

Let  $q^{(3)}$  and  $q^{(4)}$  denote the heat fluxes at the points of intersection of the lines joining the centers of the boundary elements with the side boundaries; then direct and inverse parametric fitting with respect to near-boundary elements gives

$$U'_{i1} = R'_i q^{(3)} + S'_i q^{(4)} + P'_i, \quad i = 1, \dots, I, \quad (13)$$

and parametric interpolation of the given expressions gives

$$U'_{r1} = R'_r q^{(3)} + S'_r q^{(4)} + P'_r. \quad (14)$$

Expressing  $q^{(3)}$  and  $q^{(4)}$  in terms of the fluxes at the corresponding node points by parametric interpolation

$$q^{(3)} = \sum_s d_s q_s^{(3)}, \quad q^{(4)} = \sum_{\eta} d_{\eta} q_{\eta}^{(4)} \quad (15)$$

and substituting Eq. (15) into Eq. (14), it is found that

$$U'_{r1} = R'_r \sum_s d_s q_s^{(3)} + S'_r \sum_{\eta} d_{\eta} q_{\eta}^{(4)} + P'_r. \quad (16)$$

The remainder of the procedure is as above for consistent grids, except that, in calculating the temperature in the subregion, the heat fluxes at the rod boundaries are determined by interpolation with respect to their values at the node points.

Note that this method may be extended without difficulty to systems containing three-dimensional subregions.

#### NOTATION

T, U, temperature;  $\theta$ , temperature at the boundary between subregions; q, heat flux; t, time;  $x_{\alpha}$ , coordinate;  $\beta$ , dimensionality of subregion;  $c_p$ , volume specific heat;  $\lambda$ , thermal conductivity.

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#### METHOD OF DETERMINING THE THERMOPHYSICAL PROPERTIES OF ORTHOTROPIC MATERIALS FROM THE SOLUTION OF A TWO- DIMENSIONAL INVERSE HEAT-CONDUCTION PROBLEM

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UDC 536.2.08

An unsteady two-dimensional inverse coefficient problem of heat conduction is formulated mathematically and solved.

Recent years have seen the active development of methods of determining the thermophysical properties (TPC) of materials which make use of empirical data obtained from the unsteady heating of specimens [1-3]. The theoretical foundation of these methods are mathematical formulations of unsteady inverse coefficient problems of heat transfer, which are usually inverse heat-conduction problems (ICP). The overwhelming majority of ICP mathematical formulations are based on the assumption that heat transfer is unidimensional - an assumption which is keeping investigators from making thermophysical studies more informative and applicable to a broader range of temperatures. This is particularly true in regard to comprehensive study of the TPC of anisotropic materials, the use of concentrated energy flows for heating materials, and study of TPC directly on objects of complex structure and shape. The practical resolution of these issues - which will mark a new step in the methodology of thermophysical studies - should begin with the mathematical formulation and solution of unsteady multidimensional ICP.

We will examine the mathematical formulation and solution algorithm of a two-dimensional nonlinear coefficient ICP. Let the object in thermal tests be a flat rectangular specimen made of a homogeneous orthotropic material in which the principal axes of the thermal conductivity tensor coincide with the coordinate axes  $x_1$  and  $x_2$ . The TPC of the material - the volumetric heat capacity  $c_p$  and the thermal conductivities  $\lambda_{x1}$ ,  $\lambda_{x2}$  - are dependent on temperature. The initial temperature of the specimen and the heat-transfer conditions on its faces are known. During heating (cooling), temperature is measured at several points of the

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Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 56, No. 3, pp. 483-491, March, 1989. Original article submitted April 18, 1988.